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IDENTIFICATION OF MOVING MAGNETIC FIELD LINES II; APPLICATION TO A MOVING, NON-RIGID CONDUCTOR

FRANK C. JONES
THOMAS J. BIRMINGHAM

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GSFC
GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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Frank C. Jones and Thomas J. Birmingham
NASA-Goddard Space Flight Center, Greenbelt, Maryland

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The restriction to a stationary, conducting surface in the proof of Birmingham and Jones (1968) is removed. When the conducting surface S is moving and/or deforming, field lines are identified by the fixed material point at which they pierce S, provided $E_{\parallel}$ is zero everywhere outside S. We illustrate the utility of this field line identification by examples in which general results follow from knowing the topology of the magnetic field.

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Introduction

In a previous paper (Birmingham and Jones, 1968) hereafter called I, we condidered the identification of magnetic field lines, all of which at least once pierce a stationary, arbitrarily shaped, perfectly conducting surface. (The motion of the field lines is due to time varying currents external to and/or within the surface.) Provided that the entire region external to the surface is filled with a conducting plasma so that no electric field parallel to $\underset{\sim}{B}$ exists, a magnetic field line is usefully identified by the fixed point (or points) at which it crosses the surface: the same field line always intersects the surface at the same point.

This identification of magnetic field lines is useful in following the motion of low energy charged particles (for which magnetic gradient and curvature drifts are unimportant). Such particles undergo $\underset{\sim}{E}\times\underset{\sim}{B}$ drifts which keep them frozen to field lines identified in this manner. Thus, for example, one can legitimately follow the motion of a low energy particle during compressions and expansions of a model magnetosphere by merely tracing the motion of the magnetic field line on which the particle is initially located (Hones, 1963; references in I).

We show in this paper that the restriction to a stationary surface in I is unnecessary. When the conducting surface is moving and/or distorting, the

useful field line identification is the one in which the same field line always cuts the surface at the same material point. The identification of field lines here, of course, reduces to that of I when the conducting surface is stationary.

Use of the field line identification introduced in this paper permits us to make general predictions about the motion of particles in the presence of moving magnets or the combined effect on particles of both moving magnets and external currents.

We stress that the results of this paper are general, untied to any particular model of the electric and magnetic fields. We require only that all magnetic field lines pierce the perfectly conducting surface at least once and that the parallel electric field be zero everywhere outside the surface.

The utility of our results is that the motion of low energy charged particles becomes evident from the mere topology of magnetic field lines. The often cumbersome task of calculating explicitly the electric field (Schulz and Eviatar, 1969) is unnecessary.

To some readers our results will seem intuitively obvious on the basis of Cowling's Theorem (Lundquist, 1951; 1952). Cowling's Theorem, however, applies to a medium in which the electrical conductivity is infinite in all directions. We assume only that the conductivity parallel to $\underset{\sim}{B}$ is infinite; the perpendicular conductivity is unspecified and may even be zero.

Field Line Identification

As in I we introduce the notion of the Euler potentials $\alpha(\underset{\sim}{r},t)$ and $\beta(\underset{\sim}{r},t)$ where $\underset{\sim}{B} = \nabla\alpha \times \nabla\beta$ and we may define the vector potential

$\underset{\sim}{A} = \alpha \underset{\sim}{\nabla} \beta$. We have for the electric field

$$\begin{aligned} \underset{\sim}{E} &= -\frac{1}{c}\frac{\partial \underset{\sim}{A}}{\partial t} - \underset{\sim}{\nabla}\Phi \\ &= \frac{1}{c}\left(\frac{\partial \beta}{\partial t}\underset{\sim}{\nabla}\alpha - \frac{\partial \alpha}{\partial t}\underset{\sim}{\nabla}\beta\right) - \underset{\sim}{\nabla}(\Phi + \Psi) \\ &= -\frac{\underset{\sim}{W} \times \underset{\sim}{B}}{c} - \underset{\sim}{\nabla}(\Phi + \Psi) \end{aligned} \tag{1}$$

where

$$\Psi = \frac{1}{c}\alpha\frac{\partial \beta}{\partial t}$$

The velocity

$$\underset{\sim}{W} = \left(\frac{\partial \beta}{\partial t}\underset{\sim}{\nabla}\alpha - \frac{\partial \alpha}{\partial t}\underset{\sim}{\nabla}\beta\right) \times \frac{\underset{\sim}{B}}{B^2} \tag{2}$$

was shown in I to fulfill the requirements of a valid flux preserving field line velocity in which the field lines move in such a way as to preserve their α, β labels.

An electric field causes low energy trapped particles to drift with a velocity

$$\underset{\sim}{v_E} = c\,\frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} \tag{3}$$

which is itself a valid field line velocity if

$$\nabla \times \left[\underset{\sim}{E} + \frac{\underset{\sim}{v_E} \times \underset{\sim}{B}}{c} \right] = 0$$

The two velocities differ by

$$\underset{\sim}{v}_E - \underset{\sim}{W} = c \; \frac{\underset{\sim}{B} \times \underset{\sim}{\nabla} (\Phi + \Psi)}{B^2} \tag{4}$$

In I we considered the case where all magnetic lines of force pass at least once through a perfectly conducting, stationary surface S. We were able to show that a given field line could be uniquely labeled with the values of α and β determined by the point (or points) where it crosses the surface S. This was possible because on a perfectly conducting surface $J\left(\frac{\alpha,\ \beta}{X_1, X_2}\right)$ = const. where J is the Jacobian and X_1 and X_2 are curvilinear coordinates in the surface. From this condition it followed that we could make the time independent gauge choice $\partial\beta/\partial t = \partial\alpha/\partial t = 0$ everywhere on the surface and the values of α and β on the surface uniquely label a field line for all time. This gauge choice is equivalent to the condition $\underset{\sim}{W} = 0$ on S.

Combining this condition with the fact that the tangential electric field must vanish at the conducting surface we see from equation (1) that $\Phi + \Psi$ is constant on the surface S. Filling the volume with a plasma such that $\underset{\sim}{E} \cdot \underset{\sim}{B} = 0$ forces $\Phi + \Psi$ to be constant along any field line and hence throughout all space. Equation (4) then says that $\underset{\sim}{v}_E = \underset{\sim}{W}$ and trapped particles' $\underset{\sim}{E} \times \underset{\sim}{B}$ drift motions cause them to remain "frozen" to a particluar field line which is defined by the fixed point where it crosses the surface S.

In the foregoing, which is a brief review of the argument presented in I, it was assumed that the surface was rigid and fixed in space. It is the purpose of this paper to show that this condition is not necessary; the surface S may be allowed to move, bend, and even change its area so long as the surface coordinates X_1 and X_2 refer to a fixed material point of the surface.

In I we showed that if $\hat{n}$ is the (fixed) unit normal to the surface at X_1, X_2 then $\hat{n} \cdot B \propto J\left(\frac{\alpha, \beta}{X_1, X_2}\right)$ and since $\hat{n} \cdot \underset{\sim}{B}$ = const. for a rigid, fixed conductor we were able to infer $J\left(\frac{\alpha, \beta}{X_1, X_2}\right)$ = const. However, if we consider a small area of the surface $dX_1 dX_2$ we have

$$J\left(\frac{\alpha, \beta}{X_1, X_2}\right) dX_1 dX_2 = d\alpha(X_1, X_2)\, d\beta(X_1, X_2) = d\phi(X_1, X_2)$$

which is the magnetic flux through the area defined by dX_1 and dX_2 (Northrop, 1963; Stern, 1966). We now note that if the material of the surface is a perfect conductor any motion of this material will be flux preserving, i.e., any closed loop of material will move in such a way as to keep the flux that it encircles constant. Thus if the coordinate system X_1, X_2 is fixed with respect to the material of the surface, an area of a particular material loop will be characterized by $dX_1\, dX_2$ where dX_1 and dX_2 are constant. Flux preserving motion then implies $J\left(\frac{\alpha, \beta}{X_1, X_2}\right) dX_1\, dX_2$ = const. and hence $J\left(\frac{\alpha, \beta}{X_1, X_2}\right)$ = const. If $\alpha(t_1)$, $\beta(t_1)$ and $\alpha(t_2)$, $\beta(t_2)$ label a fixed material (X_1, X_2) point at two times t_1 and t_2, it then follows that $J\left(\frac{\alpha(t_1), \beta(t_1)}{\alpha(t_2), \beta(t_2)}\right) = 1$,

so that (as indicated in I) changes of α and β with time at X_1, X_2 are simply relabeling or gauge transformations.

We can now see that a unique labeling of the lines of force is still possible as long as the labeling is referred to the <u>material point</u> where the field line crosses the surface. The boundary conditions on the surface are no longer $\frac{\partial\alpha}{\partial t} = \frac{\partial\beta}{\partial t} = 0$ but rather $\frac{\partial\alpha}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla}\alpha = \frac{\partial\beta}{\partial t} + \underset{\sim}{V}\cdot\underset{\sim}{\nabla}\beta = 0$ where $\underset{\sim}{V} = \underset{\sim}{V}(X_1, X_2)$ is the instantaneous velocity of the surface point X_1, X_2. Inserting this condition in equation (2) we obtain for the boundary value of $\underset{\sim}{W}$

$$
\begin{aligned}
\underset{\sim}{W}\Big|_S &= -\underset{\sim}{V}\cdot\left(\underset{\sim}{\nabla}\beta\,\underset{\sim}{\nabla}\alpha - \underset{\sim}{\nabla}\alpha\,\underset{\sim}{\nabla}\beta\right)\times\underset{\sim}{B}/B^2 \\
&= -(\underset{\sim}{V}\times\underset{\sim}{B})\times\underset{\sim}{B}/B^2 = \underset{\sim}{V} - \underset{\sim}{B}(\underset{\sim}{V}\cdot\underset{\sim}{B})/B^2 \\
&= \underset{\sim}{V}_\perp
\end{aligned}
\tag{5}
$$

The boundary value of $\underset{\sim}{W}$ on S is then just equal to $\underset{\sim}{V}_\perp$. The boundary condition on the tangential electric field is now that $\underset{\sim}{E}_{tan} = -\frac{\underset{\sim}{V}\times\underset{\sim}{B}}{c}$ so we see that we still have the condition $\Phi + \Psi$ = const. on the surface. The condition $\underset{\sim}{E}\cdot\underset{\sim}{B} = 0$ once again forces $\Phi + \Psi$ to be constant along any field line and hence constant everywhere so we are led once more to the condition $\underset{\sim}{v}_E = \underset{\sim}{W}$ everywhere and in addition $\underset{\sim}{v}_E = \underset{\sim}{W} = \underset{\sim}{V}_\perp$ on the surface S.

The reason that our result is the same as that obtained using Cowling's Theorem is due to the presence of the conducting surface. In our model perpendicular electric fields in the moving frame are short-circuited by charge

motion along field lines together with <u>charge redistribution between field lines via S</u>. This manner of short-circuiting depends on a global picture of the plasma and is in contrast with the MHD (Cowling) picture where the short-circuiting is produced <u>locally</u> by charge motions perpendicular to $\underset{\sim}{B}$.

<u>Examples</u>

We distinguish between situations in which the source of $\underset{\sim}{B}$ lies totally within S (Case a) and those where there are external sources (Case b).

For Case a magnetic field lines move rigidly with S (assuming that the Alfven, propagation speed of $\underset{\sim}{B}$ is much faster than the speed at which S is moving) and low energy particles drift with them. This conclusion is independent of the nature of $\underset{\sim}{B}$. In particular it holds for many of the familiar models: $\underset{\sim}{B}$ is the field of a centered dipole rotating about its magnetization axis (Davis, 1947, 1948); or $\underset{\sim}{B}$ is the field of a dipole, centered or eccentric, whose magnetization and rotation axes are non-parallel (Hones and Bergeson, 1965); or $\underset{\sim}{B}$ is the composite field of several multipole sources (e.g., Cain et al., 1967).

An example of Case a is the uniform rotation of S. Field lines then rotate rigidly, whatever the model. Low energy particles, frozen to field lines, also co-rotate if one neglects gyration and bounce motion. (The gyration and bounce motion produce no net change in position or velocity when averaged over a gyration and bounce period respectively.) It is clear then that a particle's drift energy does not exceed the (constant) energy it attains when its position on the rotating field line is farthest from the rotation axis. Rotating magnets are thus useless for energizing low energy

particles (Hones, 1963).

The results of the preceding two paragraphs are not surprising nor, as applied to specific models, are they new. Our claim is that as derived in this paper these results are general and rigorous.

In Case b field lines also remain entrenched in S and their feet move rigidly with S. External to S the field lines move simply, though not rigidly, with S provided that field lines are not rooted in any other conducting body. (Rooting in an external conductor can lead to such irregularities as winding of field lines about S.) If, for example, S rotates uniformly with period T , closed field lines, by symmetry, move periodically (with period T). Since low energy particles are frozen to field lines, their drifts are also T periodic. Net energization of such particles is once again impossible.

The results of the preceding paragraph apply, for example, to the earth rotating beneath any magnetospheric model in which field lines are closed (e.g., Mead, 1964). As the earth rotates, fields lines and the low energy particles on them are carried around the earth with a 24 hour periodicity. Schulz's (1970) result is thus clear without need of any mathematical analysis.

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